SLE and lattice models: towards off-criticality

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1 Loewner evolution: notation

If $\gamma : [0, \infty) \to \mathbb{H}$ is a continuous curve with no transversal self-intersections ("non self-crossing"), and $\gamma(0) \in \mathbb{R}$, write H_t for $t \geq 0$ for the unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$. Set $K_t := \mathbb{H} \setminus H_t$, the "hull" generated by $\gamma([0, t])$.

For each $t \geq 0$, $\exists$ a unique conformal isomorphism

$$g_t : H_t \to \mathbb{H}$$

(the "mapping out function") such that $g_t(z) = z + \frac{a_t}{z} + O(\frac{1}{z^2})$ as $z \to \infty$, where a_t is the so-called *half-plane capacity* of K_t. This is strictly increasing in t so we may assume that γ is parametrised such that $a_t = 2t$ for $t \geq 0$ (the half-plane capacity parameterisation).

Then we can define the *driving function*

$$\xi_t := g_t(\gamma(t))$$

(defined by a limit) and Loewner's theorem says that ξ is a continuous real-valued function, and for all $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t} dt \quad ; \quad \forall\, t \leq \tau_z := \sup\{s : g_s(z) \in \mathbb{H}.\} \tag{1}$$

We also set

$$Z_t(z) := g_t(z) - \xi_t;$$

the *centered Loewner flow*.

Loewner's theorem also has a converse which allows one, given a continuous real valued function $(\xi_t)_{t \geq 0}$, to define a family of locally-growing compact $\mathbb{H}$-hulls $(K_t)_{t \geq 0}$, the "Loewner chain", and associated conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ such that (1) holds. It is not necessarily the case that $(K_t)_{t \geq 0}$ is generated by a curve $(\gamma(t))_{t \geq 0}$.

2 The critical, non-massive case

2.1 Schramm–Loewner evolution (SLE)

For $\kappa > 0$, SLE_κ in $\mathbb{H}$ from 0 to ∞, is the (random) Loewner chain with driving function

$$\xi_t := \sqrt{\kappa} B_t$$

for B a standard linear Brownian motion. It has been shown that it *is* generated by a curve almost surely for all values of $\kappa > 0$.

Scale/conformal invariance Let γ be a non-self crossing curve in the half-plane parameterisation, and for $a > 0$ define $(\tilde{\gamma}(t))_{t\geq 0} = (a\gamma(\frac{t}{a^2}))_{t\geq 0}$. Then one can check that $\tilde{g}_t : z \mapsto ag_{t/a^2}(z/a)$ is a conformal isomorphism sending the unbounded connected component of $\mathbb{H} \setminus \tilde{\gamma}([0,t])$ to $\mathbb{H}$, and has $\tilde{g}_t(z) = z + 2t/z + O(1/z^2)$ as $z \to \infty$. This means that the driving function $(\tilde{\xi}_t)_{t\geq 0}$ of $\tilde{\gamma}$ is given by $\tilde{\xi}_t = \tilde{g}_t(\tilde{\gamma}(t)) = ag_{t/a^2}(\gamma(t)) = a\xi_{t/a^2}$ for $t \geq 0$.

So, if γ is an SLE_κ for some value of κ, then the scaled curve $\tilde{\gamma}$ has the same law as γ, by Brownian scaling. *In other words, SLE_κ is scale invariant.* Conversely, if a random non-self crossing curves from 0 to ∞ in $\mathbb{H}$ is scale invariant, then its driving function satisfies Brownian scaling.

Scale invariance also allows us to unambiguously define the law of SLE_κ in any simply connected domain D between two marked boundary points a and b. It is defined to be the conformal image of SLE_κ from 0 to ∞ in $\mathbb{H}$ under (any) conformal isomorphism sending $\mathbb{H}$ to D, 0 to a and ∞ to b.

Markov property Again, let γ be a non-self crossing curve in the half-plane parametrisation and for $s > 0$ define $(\gamma^s(t))_{t\geq 0} = g_s(\gamma(s+t))_{t\geq 0}$ and $(\tilde{\gamma}^s(t))_{t\geq 0} = \gamma^s(t) - \xi_s$ so that $\tilde{\gamma}^s(0) = 0$. If g_t^s is the mapping out function of γ^s at time t, then one can check that $g_t^s \circ g_s$ is a conformal isomorphism from $\mathbb{H} \setminus \gamma([0,t+s])$ to $\mathbb{H}$, with $z \mapsto z + 2(t+s)/z + O(1/z^2)$ as $z \to \infty$. Therefore, by uniqueness, we have $g_t^s \circ g_s = g_{t+s}$, i.e. $g_t^s = g_{t+s} \circ g_s^{-1}$ for $t \geq 0$. This means that the driving function of γ^s is given by $\xi^s(t) = g_t^s(\gamma^s(t)) = g_{s+t}(\gamma(t+s)) = \xi_{s+t}$. Since $\tilde{g}^s(z) = g^s(z+\xi_s) - \xi_s$, the driving function of $\tilde{\gamma}^s$ is given by $\tilde{\xi}_t^s = \xi_{s+t} - \xi_s$ for $t \geq 0$.

So, by the Markov property of Brownian motion, if γ is an SLE_κ curve for some $\kappa > 0$, then for any $s > 0$, $\tilde{\gamma}^s$ is independent of $\gamma([0,s])$, and has the same law.

Characterisation Combining the above two paragraphs implies the following. For any $\kappa > 0$, SLE_κ defines a family $\mu^{D,a,b}$ of laws on non-crossing curves in simply connected domains D from a boundary point a to a boundary point b, which is *conformally invariant* and satisfies a *Markov property.* More precisely:

- **(CI)** if $\varphi : D \to D'$ is a conformal isomorphism sending $a \to a'$ and $b \to b'$ then if $\gamma \sim \mu^{D,a,b}$ then $\varphi(\gamma) \sim \mu^{D',a',b'}$; and
- **(MP)** if $\gamma \sim \mu^{D,a,b}$, then conditionally on some an initial portion $\gamma([0,s])$ of γ, $(\gamma(s+t))_{t\geq 0} \sim \mu^{D\setminus\gamma([0,s]),\gamma(s),b}$.

Conversely, if any such family of laws satisfies **(CI)** and **(MP)** then there exists $\kappa > 0$ such that $\mu^{D,a,b}$ is the law of SLE_κ from a to b in D for every D, a, b.

Scaling limits This is why Oded Schramm introduced SLE: this characterisation makes them the only possible candidates for scaling limits of certain interfaces in statistical mechanics models at their critical point, which are at least conjectured to satisfy **(CI)** and **(MP)**. For example, critical percolation, critical Ising model, critical FK cluster model etc. with Dobrushin boundary conditions.

2.2 Conformally covariant martingale observables

Problem: it's usually pretty hard to show conformal invariance of the limiting interfaces (even if there are convincing arguments in the physics literature).

Idea: one successful idea that is present in many works by now, but outlined for example in Smirnov's ICMS proceedings article, is that proving *one* observable converges and satisfies analogues of **(CI)** and **(MP)** in the limit (for every possible configuration D, a, b) can be enough to identify a scaling limit.

Example: Percolation Take site percolation on the an ε-scale triangular lattice approximation D_ε to a domain D with two marked boundary points a and b, and open boundary conditions on the clockwise arc ab, closed boundary conditions on the anti-clockwise arc ba. Denote its law by $\mathbb{P}_\varepsilon$ and let γ_ε be the interface from a to b (living on the hexagonal lattice) that keeps open vertices to its left and closed vertices to its right. For x on the arc ab and y on the arc ba, set

$$p_\varepsilon(D, a, x, b, y) = \mathbb{P}_\varepsilon(\text{ there exists an open cluster from } ax \text{ to } by \text{ in } D_\varepsilon).$$

It is not too hard to see that this is the same as the probability that γ_ε hits the arc by before bx. Moreover, given $\gamma_\varepsilon([0, s])$ for some s (in some time parametrisation), the conditional probability of an open cluster joining ax to by is just the probability that $(\gamma_\varepsilon(t+s))_{t\geq 0}$ hits by before bx, if it has not already done so. This in turn is the probability that there is an open cluster in $D_\varepsilon \setminus \gamma_\varepsilon([0, s])$ joining the arc $\gamma_\varepsilon(s)x$ to by. That is,

$$\begin{aligned} &\mathbb{P}_\varepsilon(\text{ there exists an open cluster from } ax \text{ to } by \text{ in } D_\varepsilon \,|\, \gamma_\varepsilon([0, s])) \\ &\qquad = \mathbb{P}_\varepsilon(\text{ there exists an open cluster from } \gamma_\varepsilon(s)x \text{ to } by \text{ in } D_\varepsilon \setminus \gamma_\varepsilon([0, s]). \qquad (2) \end{aligned}$$

Now, it is actually possible to prove that

$$p_\varepsilon(D, a, x, b, y) \to p(D, a, x, b, y)$$

as $\varepsilon \to 0$ for any D, a, b, x, y, where p is an explicit function (in some domain) that is conformally invariant. Putting this together with (2) implies that

$$p(D \setminus \gamma([0, t]), \gamma(t), b, x, y) \text{ is a martingale}$$

for any scaling limit γ of γ_ε (from a to b in D) as $\varepsilon \to 0$, and any D, a, b, x, y. Indeed, if $\mathcal{F}_s = \sigma(\gamma(u); u \leq s)$ for $s > 0$ then for $t \geq s$

$$\begin{aligned} \mathbb{E}[p(D \setminus \gamma([0, t]), \gamma(t), x, b, y)|\mathcal{F}_s] &= \lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon(\mathbb{P}_\varepsilon(\text{ there exists an open cluster from } ax \text{ to } by \text{ in } D_\varepsilon|\mathcal{F}_t)|\mathcal{F}_s)) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon(\mathbb{P}_\varepsilon(\text{ there exists an open cluster from } ax \text{ to } by \text{ in } D_\varepsilon)|\mathcal{F}_s)) \\ &= p(D \setminus \gamma([0, s]), \gamma(s), x, b, y) \end{aligned}$$

as required.

In particular, if γ is the scaling limit of a percolation interface from 0 to ∞ in $\mathbb{H}$ then

$$p(\mathbb{H} \setminus \gamma([0, t]), \gamma(t), x, \infty, y) = p(\mathbb{H}, 1 - \frac{g_t(x) - \xi_t}{g_t(x) - g_t(y)}, 1, \infty, 0) = F(\frac{g_t(x) - \xi_t}{g_t(x) - g_t(y)})$$

is a martingale (the equality coming from conformal invariance applying the map $z \mapsto (g_t(z) - g_t(y))/(g_t(x) - g_t(y))$), where

$$F(u) := p(\mathbb{H}, 1 - u, 1, \infty, 0)$$

is Cardy's hypergeometric function. The explicit form of F then allows one to conclude that the driving function of γ, ξ, must be equal to $\sqrt{6}$ times a standard Brownian motion. This can be deduced using stochastic calculus, or more directly, just using Taylor expansion, by proving that ξ_t is a martingale with quadratic variation $6t$.

General case $F(D, a, b, c, \dots)$ is a conformally covariant martingale for SLE_κ if

$$F(D, a, b, c, \dots) = F(\phi(D), \phi(a), \phi(b), \phi(c), \dots)\phi'(b)^\beta \phi'(c)^\gamma \dots$$

for some exponents $\beta, \gamma, \dots$ and

$$F(D \setminus \gamma([0,t]), \gamma(t), b, c, \dots)$$

is a martingale. It is conformally invariant if all of the exponents $\beta, \gamma, \dots$ vanish. In particular, this means that $F(\mathbb{H}, 0, \infty, g_t(x) - \xi_t, \dots)g_t'(x)^\beta \dots$ is a martingale for SLE_κ in $\mathbb{H}$ from 0 to ∞.

If one can identify a discrete martingale for interfaces in some lattice model, and prove that it has a conformally covariant martingale in the scaling limit, of the form F above with F explicit, then there is hope to uniquely identify the limiting curve as an SLE (using similar techniques to in the percolation case).

There are various ways to try and do this, for example:

- start with a discrete object (related to some kind of probability) that has the martingale property built in, and try to establish conformal covariance in the limit;
- start with a discretisation of something that is an SLE martingale (see example below), and try to prove the discrete martingale property by connecting it to something in the lattice model.

Example: BVP For $\kappa > 0$

$$M_t^{\kappa,\beta,\sigma} = Z_t(z)^\beta Z_t'(z)^\sigma \text{ with } \sigma = \beta + \frac{\beta(\beta-1)}{4}\kappa$$

is a *holomorphic* martingale for SLE_κ in $\mathbb{H}$ from 0 to ∞ (solution of a Riemann-Hilbert problem in H_t).

It is $(dz)^\sigma$-covariant. To see that this is a martingale, note that

$$Z_t'(z) := \frac{\partial}{\partial z} Z_t(z) = \frac{\partial}{\partial z}\big(z + \int_0^t \frac{Z_s(z)}{d} s + \xi_t\big) = 1 - \int_0^t \frac{2Z_s'(z)}{Z_s(z)^2} ds$$

so

$$dZ_t'(z) = -\frac{2Z_t'(z)}{Z_t(z)^2} dt.$$

Itô's formula gives

$$d(Z_t'(z)^\sigma) = -2\sigma \frac{2Z_t'(z)^\sigma}{Z_t(z)^2} dt$$

and

$$d(Z_t(z)^\beta) = \beta Z_t(z)^{\beta-1}\Big(\frac{2}{Z_t(z)} dt - \sqrt{\kappa} dB_t\Big) + \frac{1}{2}\kappa\beta(\beta-1) Z_t(z)^{\beta-2} dt = Z_t(z)^{\beta-2}\Big(2\beta + \frac{\kappa\beta(\beta-1)}{2}\Big)dt - \sqrt{\kappa}\beta Z_t(z)^{\beta-1} dB_t$$

and therefore, since $Z_t'(z)^\sigma$ is of bounded variation,

$$dM_t^{\kappa,\beta,\sigma}(z) = Z_t'(z)^\sigma Z_t(z)^{\beta-2}\Big(2\beta - \frac{\kappa\beta(\beta-1)}{2} - 2\sigma\Big)dt - Z_t'(z)^\sigma \sqrt{\kappa}\beta Z_t(z)^{\beta-1} dB_t.$$

The drift term indeed vanishes if $\sigma = \beta + (\kappa/4)\beta(\beta - 1)$. Conversely, one can show that if an appropriately nice random curve has $M_t^{\kappa,\beta,\sigma}$ as a martingale, then the curve must be an SLE_κ.

Another example of a martingale for SLE_κ is

$$\mathbf{M}_t^\kappa(z) = \log Z_t(z) + (1 - \frac{\kappa}{4}) \log Z_t'(z)$$

or its imaginary part $M_t^\kappa(z) = \arg Z_t(z) + (1 - \kappa/4) \arg Z_t'(z)$ which satisfies

$$\Delta M_t^\kappa(z) = 0 \text{ in } H_t \quad ; \quad M_t^\kappa(x) = \arg Z_t(x) + (1 - \kappa/4) \arg Z_t'(x) \text{ on } \partial H_t.$$

Question: M-S seems to suggest that these are the only *holomorphic* martingale observables for SLE_κ. To what extent is this true?!

3 The massive case

Two approaches to massive SLE

- Generalise the holomorphic martingale observables above to massive versions: i.e. satisfying

$$\bar{\partial} M_t^{(m),\kappa,\beta,\sigma} - \mathrm{i} m \bar{M}_t^{(m),\kappa,\beta,\sigma} = 0 \text{ or} \tag{3}$$

$$\Delta M_t^{(m),\kappa} - m^2 M_t^{(m),\kappa} = 0 \tag{4}$$

  in H_t, with the same boundary conditions as in the non-massive case. Here m, the *mass*, is a function of z. Then ask, if there is a (unique) random Loewner evolution in $\mathbb{H}$ from 0 to ∞ (with driving function $= \sqrt{\kappa} B_t$+drift) such that the massive version $M_t^{(m),\kappa,\beta,\sigma}$ or $M_t^{(m),\kappa}$ is a martingale.

- Consider scaling limits of interfaces in off-critical (massive perturbations of) lattice models, where the peturbed parameters approach criticality at the correct rate as the mesh size of the lattice goes to 0, in order to get something non-trivial. These scaling limits should be related to massive field theories, where the mass is related perturbation.

You can then ask if the two things define the same limiting random curves. For example, if there is some lattice model for which the discrete interfaces have a discrete version of $M^{\kappa,\beta,\sigma}$ or M^κ as a martingale, then it is natural to expect that an appropriate off-critical version of the lattice model will have an interface scaling limit for which the massive version of $M^{\kappa,\beta,\sigma}$ or M^κ is a martingale.

Conformal Covariance If (3) or (4) (for some κ, β) is a martingale for some random curve γ in $\mathbb{H}$ from 0 to ∞, then if $\tilde{\gamma}(t) = a\gamma(t)$ is a scaled version, it is easy to check that (3) or (4) with m replaced by $\frac{1}{a} m(\frac{\cdot}{a})$ will be a martingale for $\tilde{\gamma}$. Equivalently, if you take a conformal image $\varphi(\gamma)$, then the analogue of (3) or (4) in $\varphi(H_t)$ with m replaced by $|(\varphi^{-1})'|(m \circ \varphi^{-1})$ will be a martingale for $\varphi(\gamma)$.

So suppose that for every *(in some sense!)* mass there is a unique random curve such that (3) or (4) (for some fixed κ, β) is a martingale. Call its law $\mu^{(m),D,a,b}$. Then there will be, for each mass, and each simply connected domain D and marked boundary points a, b, a unique random curve from a to b in D such that the analogue of (3) or (4) is a martingale. Call its law $\mu^{(m),D,a,b}$. Then these laws will satisfy conformal covariance:

$$\text{if } \gamma \sim \mu^{(m),D,a,b} \text{ then } \varphi(\gamma) \sim \mu^{(|(\varphi^{-1})'|(m\circ\varphi^{-1})),\varphi(D),\varphi(a),\varphi(b)}.$$

This is also expected for scaling limits of off-critical (massive perturbations) of critical lattice models *(in an appropriate sense).*

Example Massive harmonic explorer (→ Léonie).

Questions

- Describe the whole range of "massive SLE" such that the massive martingale observables (4) and (3) are martingales. Are their driving functions always SLE plus drift? Are they always absolutely continuous with respect to SLE_κ for some κ? (Maybe *no* for $\kappa > 4$). Do they always have the same scaling exponents?

- Prove convergence of off critical lattice interface models to massive SLE. Are these limits always in the family above?

- Suppose you have a family $\mu^{(m),D,a,b}$ of laws on curves for each simply connected domain D, boundary points a, b, and mass (m) on D (satisfying something?). Can we characterise such families of laws satisfying conformal covariance **(CC)** and the Markov property **(MP)**:

  (CC) if $\varphi : D \to D'$ is a conformal isomorphism then

  $$\gamma \sim \mu^{(m),D,a,b} \Rightarrow \varphi(\gamma) \sim \mu^{(|(\varphi^{-1})'|(m\circ\varphi^{-1})),\varphi(D),\varphi(a),\varphi(b)}$$

  ; and

  (MP) if $\gamma \sim \mu^{(m),D,a,b}$, then conditionally on some an initial portion $\gamma([0,s])$ of γ, $(\gamma(s+t))_{t\geq 0} \sim \mu^{(m),D\setminus\gamma([0,s]),\gamma(s),b}$,

  similarly to ordinary SLE? How does this relate to the family in the first question?

- Can we define loop variants of massive SLE? Can they be characterised similarly to ordinary CLE?