CONFORMAL LOOP ENSEMBLES: THE MARKOVIAN CHARACTERIZATION

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Heavily based on [2] with a lot of direct quotes Mathpix.

Contents

1. Introduction

Important properties of SLE

- Conformal invariance
- Markov property

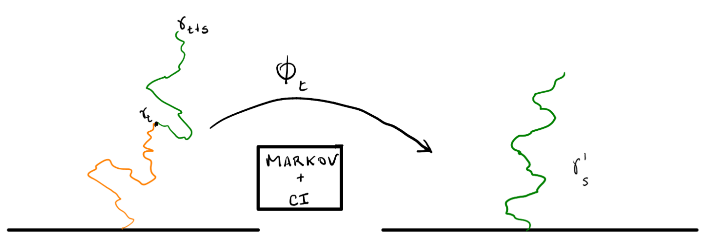

Figure 1. Basic SLE properties

The objective:

- Describe what the Markov property means in the setting of CLE
- Characterise an one-parameter family of loops that "locally looks like" SLE's but are closed loops. Once again, this family should describe certain interfaces of critical models in statistical mechanics. Notice that we are describing the whole family, not a single interface (like in SLE).

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CLE: "random collections of loops that combine conformal invariance and a natural restriction properties (motivated by the fact that the discrete analog of this property trivially holds for the discrete models we have in mind)."

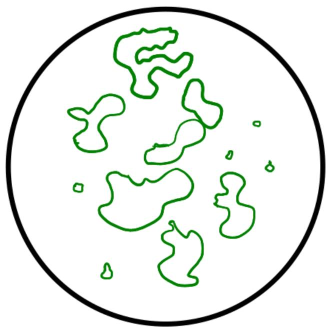


FIGURE 2. A CLE depiction

What we won't do: Provide its construction via loop soups. To see such construction in detail, see part 2 of [2]. We will also come back to this when we look at [1].

The quick idea is that we sample a collections of loops according to a Poisson point process in the space of Loops according to some intensity measure related to the Brownian Motion. Then, we consider the clusters formed by loops that intersect each other. Finally, we look at the topological boundary of the infinite connected component obtained by removing all loops in a given cluster from $\mathbb{R}^2$. This will give us *one* loop for the CLE configuration.

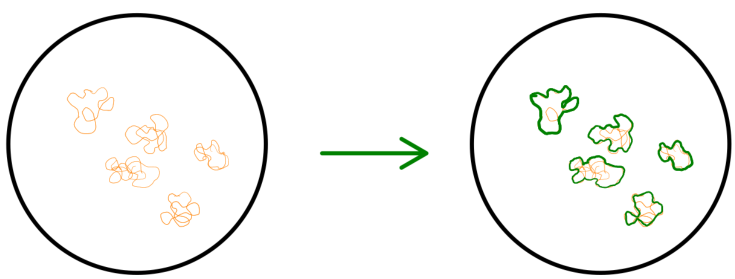


FIGURE 3. Loop soup construction of CLE

In particular, we will not prove the following theorem:

Theorem 1.

- *For each CLE, there exists a value* $\kappa \in (8/3, 4]$ *such that with probability one, all loops of the CLE are* SLE_κ*-type loops.*
- *Conversely, for each* $\kappa \in (8/3, 4]$*, there exists exactly one CLE with* SLE_κ *type loops.*

This theorem is proved as

(1) Uniqueness of the CLE: this uses the Markovian property and SLE arguments
(2) Existence of the CLE: this is the construction via loop soups.

We will look at the first part of the argument.

Although the conformal invariance has a clear analogue in the case of CLE, how does we set an analogue of the Markov property? We do not have a "past of the path" any more.

2. Proper definitions

$\Gamma = (\gamma_j, j \in J)$: Random families of **non-nested simple disjoint loops** in simply connected domains.

D: simply connected - let P_D denote the law of this loop-ensemble in D.

Non-triviality: there exists at least one loop almost surely.

Conformally invariance: if for any two simply connected domains D and D' (which are not equal to the entire plane) and conformal transformation $\psi : D \to D'$, the image of P_D under ψ is $P_{D'}$.

Local finiteness assumption: if D is equal to the unit disc $\mathbb{U}$, then for any $\varepsilon > 0$, there are $P_{\mathbb{U}}$ almost surely only finitely many loops of radius larger than ϵ in Γ.

Restriction property: $D_1 \subset D_2$: simply connected domains. Sample a family $(\gamma_j, j \in J)$ according to P_{D_2}. Then, we can subdivide the family $\Gamma = (\gamma_j, j \in J)$ into two parts:

- $(\gamma_j, j \in J_1)$ stay in D_1. Let $\Gamma' = \Gamma \setminus (\gamma_j, j \in J) \setminus (\gamma_j, j \in J_1)$
- Let $D_2^* = D \setminus (\Gamma' \cup \mathrm{int}\,\Gamma'$. We say that the family P_D satisfies restriction if, for any such D_1 and D_2, the conditional law of $(\gamma_j, j \in J)$ given D_1^* is $P_{D_1^*}$ (product of P_D for each connected component D of D_1^*).

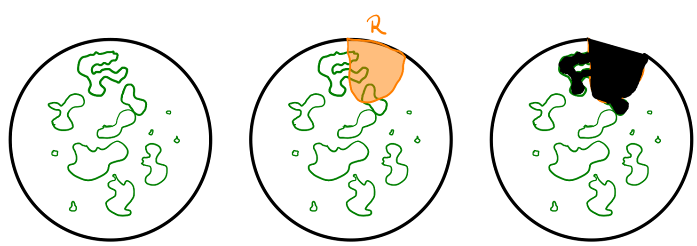


Figure 4. Restriction property

CLE = **NT** + **CI** + **LF** + **RP**.

Remark 2 (Full/nested CLE)**.** *Thanks to the CI, we can construct embedded a CLE inside each loop of the CLE. Then by iterating this process infinitely, we have the full CLE/nested CLE.*

This type of family is given by studying all *interfaces of a statistical mechanics model, rather than only the outermost ones.*

Remark 3 (Bubble measure)**.** *The relation with SLE can be made formal by looking at the so called "bubble measure". That is, for $z \in \mathbb{H}$ (due to CI, we can do this in any other domain), we look at the measure obtained by the limit of the law of the unique loop that surrounds z conditioned that such loop intersects a ε-neighbourhood of the origin. It turns out that this measure can be obtained via a limit of measures of SLEκ curves for $\kappa \in (8/3, 4]$ and such measures are distinct from each other. The existence of $\gamma(z)$ will be discussed in the next section.*

Before we proceed, we will just fix some notation.

- D: domain which is not the full plane;
- $\mathcal{L}$: space of simple loop families contained in D;
- Σ: σ-field generated by events $[O \subset \text{int}(\gamma)]$ where O is an open set. This is the same as the σ-field generated by $[x \in \text{int}(\gamma)]$ where x spans a countable and dense set.
- $\Gamma = (\gamma_j, j \in J)$ (at most countable) collection of simple loops. Let
$$\mu_\Gamma = \sum_{j \in J} \delta_{\gamma_j}.$$
We can then look at σ-field generated by $[\Gamma : \mu_\Gamma(A) = k]$ where $A \in \Sigma$ and $k \geq 0$.

3. Basic Properties

Lemma 4. *Then, for any given $z \in \mathbb{U}$, there almost surely exists a loop γ_j in Γ such that $z \in \text{int}(\gamma_j)$.*

Proof. Define $u = u(z)$ to be the probability that z is in the interior of some loop in Γ. By Moebius invariance, this quantity u does not depend on z.

Furthermore, since $P(J \neq \emptyset) > 0$, it follows that $u > 0$ (otherwise the expected area of the union of all interiors of loops would be zero). Hence, there exists $r \in (0, 1)$ such that with a positive probability p, the origin is in the interior of some loop in Γ that intersects the slit $[r, 1]$ (we call A this event).

We now define $U = \mathbb{U} \setminus [r, 1)$ and apply the restriction property. If A holds, then the origin is in the interior of some loop of Γ. If A does not hold, then the origin is in one of the connected components of $\tilde{U}$ and the conditional probability that it is surrounded by a loop in this domain is therefore still u. Hence, $u = p + (1-p)u$ so that $u = 1$. □

Corollary 5. *J is almost surely infinite.*

Proof. Almost surely, all the points $1 - 1/n, n \geq 1$ are surrounded by a loop, and any given loop can only surround finitely many of these points (because it is at positive distance from $\partial\mathbb{U}$). □

Lemma 6. *Let $M(\theta)$ denote the set of configurations $\Gamma = (\gamma_j, j \in J)$ such that for all $j \in J$, the radius $[0, e^{i\theta}]$ is never locally "touched without crossing" by γ_j (in other words, θ is a local extremum of none of the $\arg(\gamma_j)$'s). Then, for each given θ, Γ is almost surely in $M(\theta)$.*

Lemma 7. *For any $r < 1$, the probability that $r\mathbb{U}$ is entirely contained in the interior of one single loop is positive.*

Proof. This is because each simple loop γ that surrounds the origin can be approximated "from the outside" by a loop η on a grid of rational meshsize with as much precision as one wants. This implies in particular that one can find one such loop η in such a way that the image of one loop γ in the CLE under a conformal map from $\mathrm{int}(\eta)$ onto $\mathbb{U}$ that preserves the origin has an interior containing $r\mathbb{U}$. Hence, if we apply the restriction property to $U = \mathrm{int}(\eta)$, we get readily that with positive probability, the interior of some loop in the CLE contains $r\mathbb{U}$. Since this property will not be directly used nor needed later in the paper, we leave the details of the proof to the reader. □

Lemma 8. *The restriction property continues to hold if we replace the simply connected domain $U \subset \mathbb{U}$ with the union U of countably many disjoint simply connected domains $U_i \subset \mathbb{U}$. That is, we still have that the conditional law of $(\gamma_j, j \in J^*)$ given U^* (or alternatively given the family $(\gamma_j, j \in I)$) is P_{U^*}.*

Proof. To see this, note first that applying the property separately for each U_i gives us the marginal conditional laws for the set of loops within each of the U_i. Then, observe that the conditional law of the set of loops in U_i is unchanged when one further conditions on the set of loops in $\cup_{i' \neq i} U_{i'}$. Hence, the sets of loops in the domains $U_1^*, \dots, U_i^*, \dots$ are in fact independent (conditionally on $(\gamma_j, j \in I)$). □

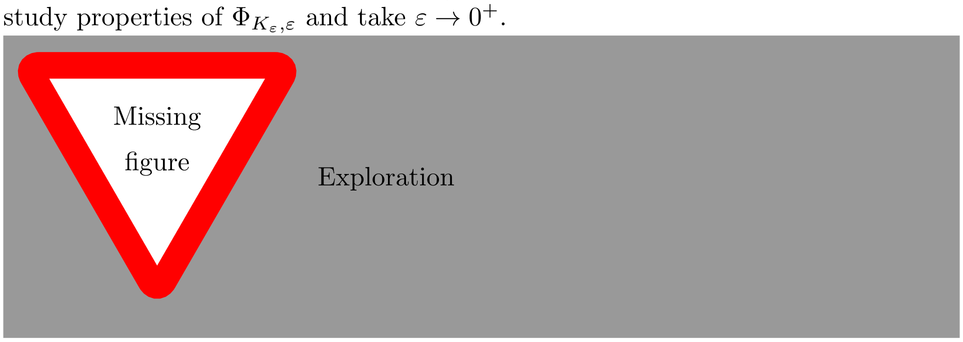
4. Explorations!

In order to understand the CLE configuration in a "Markovian way", we will discuss a procedure of explorations of loops. Let us focus on the case of the domain $D = \mathbb{U}$.

An exploration will be obtained by iterating a process. First, we will fix a parameter $\varepsilon > 0$ small. Then, we select some form of "target" for when the exploration should stop. At each step j, we select a value (randomly or deterministically) $y_j \in \partial\mathbb{U}$ and will obtain the set $U'_{j,\varepsilon} = U_{j,\varepsilon} \setminus \Phi_{j,\varepsilon}^{-1}(D(y_j, \varepsilon))$. Then define U_j as a specific choice of connected component of U_j after removing all of the loops that do not stay inside of U'_j.

Then, we define a conformal map $\Phi_{j,\varepsilon} : \mathbb{U}_{j,\varepsilon} \setminus U_{j,\varepsilon} \to \mathbb{U}$ with some constrain (say $\Phi_{j,\varepsilon}(0) = 0$ and $\Phi'_{j,\varepsilon}(0) > 0$). We then iterate the process until either we cannot define $\Phi_{j,\varepsilon}$ or (often equivalent) we have reached our target.

If we reach our target in a finite (random) number K_ε of steps, we can even study properties of $\Phi_{K_\varepsilon,\varepsilon}$ and take $\varepsilon \to 0^+$.



(1) Exploring a fixed set

(2) Exploring sequence of fixed set
(3) Exploring the loop of $\gamma(0)$
   - This provides a simpler way to study the "bubble measure"
(4) Chordal exploration on $\mathbb{H}$
(5) Radial exploration on $\mathbb{H}$

5. One-point pinned measures

We can use the third type of exploration we described above to essentially explore the line $[0,1]$ "from 1 to 0".

That is, let

$$R = \max\{r \in [0,1], r \in \gamma_0\},$$

and consider the $\hat{U}$ to be the connected component that that includes 0 once we remove from $\mathbb{U}$ the segment $[r,1]$ and the interior of all loops that intersect it.

Now, consider the exploration of the loop $\gamma(0)$ (the loop of 0), that is, take $y_j = 1$ for all j and define the U_j to be the connected component of $\mathbb{U}$ that contains $\gamma(0)$. Then define Φ_j as the map that maps U_j to $\mathbb{U}$ such that $\Phi_j(0) = 0$ and $\Phi'_j(0) > 0$. Let N be the random variable that describes the maximum j for which the map Φ_j is well defined.

Notice that due to **RP** and **CI**, we have that

$$\mathbb{P}(N \geq n) = \mathbb{P}(\gamma(0) \cap D(1,\varepsilon) = \emptyset)^n.$$

and that $\Phi_N(\gamma(0))$ is distributed according to the conditional law of $\gamma(0)$ given that $\gamma \cap D(1,\varepsilon) \neq \emptyset$.

Proposition 9. *The exploration procedure of the loop $\gamma(0)$ given above (Φ_N, y_N) converge almost surely to the pair $(\hat{y}^{-1}\hat{\Phi}, 1)$ as $\varepsilon \to 0$.*

The let μ^i be the weak limit of the measure $\mathbb{P}(\gamma(i)|\gamma(i) \cap D(0,\varepsilon))$ as $\varepsilon \to 0$ We have Let us denote by $u(\varepsilon)$ the probability that the loop $\gamma(i)$ intersects the disc of radius ε around the origin in a CLE.

- at least for almost all λ sufficiently close to one, the loop that surrounds i also surrounds $i\lambda$ and i/λ with probability at least $1/2$ under μ^i, and
- i/λ as well as λi are almost surely not on $\gamma(i)$ (when λ is fixed).

Let O_i denote the interior of the loop $\gamma(i)$. We know that

$$\lim_{\varepsilon \to 0} \frac{P(\lambda i \in O_i \text{ and } \gamma(i) \cap C_{\lambda\varepsilon} \neq \emptyset)}{u(\lambda\varepsilon)} = \mu^i(\lambda i \in O_i).$$

On the other hand, the scaling property of the CLE shows that when $\varepsilon \to 0$,

$$\begin{aligned} \frac{P(\lambda i \in O_i \text{ and } \gamma(i) \cap C_{\lambda\varepsilon} \neq \emptyset)}{u(\lambda\varepsilon)} &= \frac{P(i \in O_{\lambda i} \text{ and } \gamma(\lambda i) \cap C_{\lambda\varepsilon} \neq \emptyset)}{u(\lambda\varepsilon)} \\ &= \frac{P(i/\lambda \in O_i \text{ and } \gamma(i) \cap C_{\varepsilon} \neq \emptyset)}{u(\varepsilon)} \times \frac{u(\varepsilon)}{u(\lambda\varepsilon)} \\ &\sim \mu^i(i/\lambda \in O_i) \times \frac{u(\varepsilon)}{u(\lambda\varepsilon)}. \end{aligned}$$

Hence, for all λ sufficiently close to 1 , we conclude that

$$f(\lambda) := \lim_{\varepsilon \to 0} \frac{u(\lambda\varepsilon)}{u(\varepsilon)} = \frac{\mu^i(i/\lambda \in O_i)}{\mu^i(\lambda i \in O_i)}$$

is well defined. Furthermore, given λ, λ' and that $f(\lambda\lambda') = f(\lambda)f(\lambda')$. and $f(\lambda) \to 1$ as $\lambda \to 1$. Therefore

Proposition 10. *There exists $\beta \in \mathbb{R}$ such that*

$$f(\lambda) = \lambda^\beta.$$

Corollary 11. $u(\varepsilon) = \varepsilon^{\beta+o(1)}$ *as* $\varepsilon \to 0+$.

Because of scaling, we can now define, for all $z = i\lambda$, a measure μ^z on loops $\gamma(z)$ that surround z and touch the real line at the origin as follows:

$$\mu^z(\gamma(z) \in \mathcal{A}) = \lambda^{-\beta}\mu^i(\lambda\gamma(i) \in \mathcal{A})$$

for any measurable set $\mathcal{A}$ of loops. This is also the limit of $u(\varepsilon)^{-1}$ times the law of $\gamma(z)$ in a CLE, restricted to the event $\{\gamma(z) \cap C_\varepsilon \neq \emptyset\}$.

Let us now choose any z in the upper half-plane. Let $\psi = \psi_z$ now denote the Möbius transformation from the upper half-plane onto itself with $\psi(z) = i$ and $\psi(0) = 0$. Let $\lambda = 1/\psi'(0)$. Clearly, for any given $a > 1$, for any small enough ε, the image of C_ε under ψ is "squeezed" between the circles $C_{\varepsilon/a\lambda}$ and $C_{a\varepsilon/\lambda}$. It follows readily (using the fact that $f(a) \to 1$ as $a \to 1$) that the measure μ^z defined for all measurable $\mathcal{A}$ by

$$\mu^z(\gamma(z) \in \mathcal{A}) = \lambda^{-\beta}\mu^i\left(\psi^{-1}(\gamma(i)) \in \mathcal{A}\right)$$

can again be viewed as the limit when $\varepsilon \to 0$ of $u(\varepsilon)^{-1}$ times the distribution of $\gamma(z)$ restricted to $\{\gamma(z) \cap C_\varepsilon \neq \emptyset\}$.

Finally, we can now define our measure μ on pinned loops.

- A measure (not a probability measure) on simple loops that touch the real line at the origin and otherwise stay in the upper half-plane (this is what we call a pinned loop)
- For all $z \in \mathbb{H}$, it coincides with μ^z on the set of loops that surround z.
  - Indeed, the previous limiting procedure shows immediately that for any two points z and z', the two measures μ^z and $\mu^{z'}$ coincide on the set of loops that surround both z and z'.
- The requirement that μ coincides with the μ^z 's (as described above) fully determines μ.
- For any conformal transformation ψ from the upper half-plane onto itself with $\psi(0) = 0$, we have

$$\psi \circ \mu = |\psi'(0)|^{-\beta}\,\mu.$$

  This is the conformal covariance property of μ. Note that the maps $z \mapsto -za/(z-a)$ for real $a \neq 0$ satisfy $\psi'(0) = 1$ so that μ is invariant under these transformations.
- For each z in the upper half-plane, the mass $\mu(\{\gamma : z \in \text{int}(\gamma)\})$ is finite and equal to $\psi'(0)^\beta$, where ψ is the conformal map from $\mathbb{H}$ onto itself with $\psi(0) = 0$ and $\psi(z) = i$.
- - For each z in the upper half-plane, the measure μ restricted to the set of loops that surround z is the limit as $\varepsilon \to 0+$ of $u(\varepsilon)^{-1}$ times the law of $\gamma(z)$ in a CLE restricted to the event $\{\gamma(z) \cap C_\varepsilon \neq \emptyset\}$.

A few things we won't have time to properly go over. A priori estimates for the pinned measure. Let us denote the "radius" of a loop γ in the upper half-plane by

$$R(\gamma) = \max\{|z| : z \in \gamma\}$$

Lemma 12. *The scaling exponent β described above lies in $(0, 2)$.*

Lemma 13. *The μ-measure of the set of loops with radius greater than 1 is finite.*

This allows for scaling arguments that check the decay of $R(\gamma)$. This will be important to classify exactly which κ is necessary to recover this β.

Points to complete

(1) Two-point pinned loop probability measure
(2) SLE_κ bubble measure
(3) From Two-point to SLE
(4) From bubble measure back to CLE

References

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[2] S. Sheffield and W. Werner. Conformal loop ensembles: the markovian characterization and the loop-soup construction. *Annals of Mathematics*, pages 1827–1917, 2012.